

NON-CONTRACTIBLE ACYCLIC NORMAL SPINES

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1. Introduction

In [3], we have defined fake surfaces to study 3-manifolds with boundary from their spines. We use the notations in [3] and [4], for example, $\mathcal{F}(s, t)$ denotes the set of all the acyclic closed fake surfaces P with $\#\mathcal{S}_2(P)=s$ and $\#\mathcal{S}_3(P)=t$, where $\mathcal{S}_i(P)$ means the i -th singularity of P and $\#$ denotes the number of the connected components. And, $\mathcal{E}(s, t)$ is the subset of $\mathcal{F}(s, t)$ each of whose elements is a normal spine, that is, for any element P of $\mathcal{E}(s, t)$, there exists a 3-manifold in which P can be embedded as a spine. The following theorems are proved in [3] and [4].

Theorem. $\mathcal{F}(s, t)=\phi$, if and only if $t=0$.

Theorem. $\mathcal{E}(s, t)=\phi$, if and only if $s \geq 2t$.

Then, when $t \geq 1$, it is known that the difference $\mathcal{F}(s, t) - \mathcal{E}(s, t)$ is non-empty.

Let $\mathcal{C}(s, t)$ denote the subset of $\mathcal{E}(s, t)$ each of whose elements is contractible and $\mathcal{B}(s, t)$ the subset of $\mathcal{C}(s, t)$ each of whose elements is a normal spine of a 3-ball. Define the two difference sets $\mathcal{D}(s, t)$ and $\mathcal{A}(s, t)$ by

$$\begin{aligned}\mathcal{D}(s, t) &= \mathcal{E}(s, t) - \mathcal{C}(s, t), \\ \mathcal{A}(s, t) &= \mathcal{C}(s, t) - \mathcal{B}(s, t).\end{aligned}$$

Then, Poincaré conjecture asks "Is the set $\bigcup_{s,t} \mathcal{A}(s, t)$ empty?". On the other hand, the following theorem is well-known.

Theorem. $\bigcup_{s,t} \mathcal{D}(s, t) \neq \phi$.

And, in [3] and [4], we proved the following.

Theorem. $\mathcal{D}(s, t)=\phi=\mathcal{A}(s, t)$ for the cases $s=2t-1$ and $s=2t-2$, and $\mathcal{D}(1, 2)=\phi=\mathcal{A}(1, 2)$.

In this paper, we show the following.

Theorem 1. *For the case $1 \leq s \leq 2t - 11$ and $t \geq 6$, the set $\mathcal{D}(s, t)$ is non-empty.*

In § 2, we construct a non-contractible acyclic normal spine P_k with $\#\mathcal{S}_2(P_k)=1$ and $\#\mathcal{S}_3(P_k)=8k-1$ for any integer $k \geq 1$. And, in § 3, we can prove that a 3-manifold W_1 has a normal spine P' with $\#\mathcal{S}_2(P')=1$ and $\#\mathcal{S}_3(P')=6$, where W_k is the 3-manifold containing P_k as its normal spine. And, the proof of Theorem 1 is obtained. It is known, by the uniqueness theorem of [1], that W_k is uniquely determined. In § 4, we define the *Dehn space of type k* and show, in Theorem 2, that W_k is the Dehn space of type k .

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2. The construction of non-contractible acyclic normal spines P_k

It has been proved in Theorem 4 [3] that $\mathcal{E}(1, 1)$ contains a unique element $F_{1,1}^1$, called an abalone. Let the set $\{M_1, M_2, f\}$ be the polygonal representation of the abalone, that is, M_i is a 2-ball for $i=1, 2$, and f means the identification map from $M_1 \cup M_2$ to $F_{1,1}^1$ (for M_1, M_2 and the identification by f , see Theorem 2 [3]).

Through out this paper, the subpolyhedron $f(M_2)$ of the abalone is denoted by F , which is written in Fig. 1. Then, F is a closed fake surface with $\#\mathcal{S}_2(F)=1$ and $\#\mathcal{S}_3(F)=0$, more precisely, $U(F)=S \times_{\sigma} T$. And, by a little geometrical consideration, it is seen that F is a normal spine of the exterior of the clover-leaf knot in 3-sphere. The fundamental group of F is as follows.

$$\pi_1(F) = (S_1, S_2; S_1 S_2^{-1} S_1 S_2^2 = 1)$$

(for the generators S_1 and S_2 , see Fig. 1).

Lemma 1. *For any integer $k \geq 0$, there exists an embedding h_k from 1-sphere S into F which represents the homotopy class $S_2^{3k} S_1^{6k-1}$ and the intersection $h_k(S) \cap S_2$ consists of $|8k-1|$ points.*

Proof. When $k=0$, we can take h_0 to be the homeomorphism from S onto S_1 which reverses the orientation. Then, clearly, h_0 represents the homotopy class S_1^{-1} and we have $\#(h_0(S) \cap S_2)=1$. Let us construct the required embedding h_k for the cases $k \geq 1$.

Step 1. Suppose $k=1$. For the point a, a', b, b', c, c', d and $x_i, i=1, 2, 3$, see Fig. 2. Now, starting from the point a , go to b along the orientation of S_2 . From b' , go to c along S_2 . Intersecting with S_2 at the point x_1 , go to c' as shown in Fig.2. From c' , go to d along S_2 . And, intersecting with S_2 at x_2 , go to x_3 .

Step 2. Because S_2^3 lies in the center of $\pi_1(F)$, we obtain the following.

$$S_2^3 S_1^{6k-1} = (S_2^3 S_1^5) \prod_{p=2}^k (S_2^3 S_1^6)_p, \quad k \geq 2.$$

So, we try to construct the required embedding h_k to represent the homotopy class $(S_2^3 S_1^5) \prod_{p=2}^k (S_2^3 S_1^6)_p$, as follows. Let a_2, \dots, a_k be the points between a_1 and S_1 as shown in Fig. 3. And formally, set $a_1 = a$. Then, by the same way as in Step 1, we obtain an embedding h_p' from S into F which represents the homotopy class $(S_2^3 S_1^6)_p$ and whose initial point and end point is a_p . And set $h_1' = h_1$. More strictly, we can choose h_p' to satisfy the following conditions.

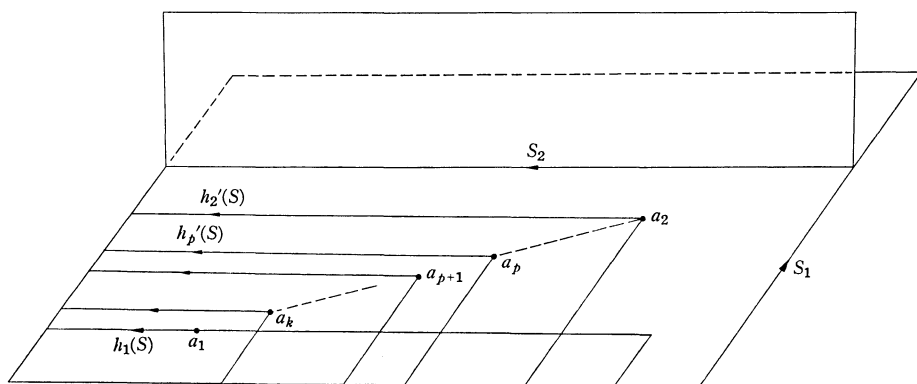


Fig. 3

- (1) $h_p'(S) \cap h_q'(S) = \emptyset$, if $p \neq q$ and $p, q \geq 2$.
- (2) $h_p'(S) \cap h_1(S)$ is one point in the small neighborhood of a_1 (see Fig. 3).
- (3) $\#(h_p'(S) \cap S_2) = 8$.

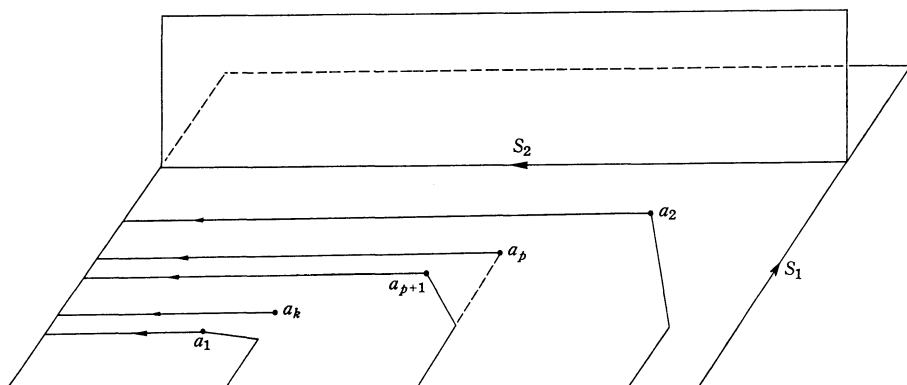


Fig. 4

Now, changing the end point a_p of h_p' to the initial point a_{p+1} of h_{p+1}' , for $1 \leq p \leq k-1$, and the end point a_k to a_1 (see Fig. 4), we obtain the required embedding h_k from S into F .

DEFINITION 1. Let P_k be the closed fake surface obtained from F by attaching a 2-ball B by the homeomorphism h_k from B to F .

REMARK. From the construction, it is clear that P_0 is homeomorphic to an abalone $F_{1,1}^1$.

Lemma 2. *If $k \geq 1$, then P_k is a non-contractible element of $\mathcal{E}(1, 8k-1)$.*

Proof. We can prove that P_k is acyclic, because

$$\begin{aligned} H_1(P_k) &= (S_1, S_2; 2S_1 + S_2 = 0 \quad (6k-1)S_1 + 3kS_2 = 0) \\ &= 0 \end{aligned}$$

and $H_2(P_k)$ is trivially trivial. And the fact that $\pi_1(P_k)$ is non-trivial follows from the calculation in [2]. Hence, P_k is a non-contractible acyclic closed fake surface. It follows from the construction of P_k that $U(P_k)$ can be embedded in the euclidean 3-space R^3 . Then, by Lemma 2 [4], P_k is a normal spine. And, again from the construction, we see $\# \mathcal{S}_2(P_k) = 1$ and $\# \mathcal{S}_3(P_k) = 8k-1$, more precisely, $\mathcal{S}_2(P_k) = S_2 \cup h_k(S)$ and $\mathcal{S}_3(P_k) = S_2 \cap h_k(S)$ is the union $\bigcup_{p=1}^k (S_2 \cap h_p'(S))$, and we obtain $\# \mathcal{S}_3(P_k) = 8k-1$.

3. The element P' of $\mathcal{D}(1, 6)$ and the proof of Theorem 1

Let W_k denote the 3-manifold containing P_k as its normal spine, $k=1, 2, \dots$. In this section, we consider P_1 in W_1 and construct another normal spine P' of W_1 from P_1 in $\mathcal{D}(1, 6)$. For the polygonal representation of P_1 , see Fig. 5.

Proposition 1. *W_1 has a normal spine P' in $\mathcal{D}(1, 6)$.*

Proof. Let us consider M_1 of the polygonal representation of P_1 , and let N be the regular neighborhood of M_1 mod \dot{M}_1 in W_1 chosen to satisfy

$$N \cap (P_1 - \dot{M}_1) = \dot{N} \cap P_1 = \dot{M}_1 \times I,$$

as shown in Fig. 6, where I is the closed unit interval $[0, 1]$ and $M_1 = M_1 \times 1/2$. Put $A = \dot{N} \cap P_1$. Then, $A = (A \cap \mathcal{S}_2(P_1))$ has three connected components each of whose closures is a 2-ball. Take such a 2-ball B . Regarding B as a free face of $P_1 \cup N$, we can collapse $P_1 \cup N$ to $(P_1 - (N \cap P_1)) \cup (\dot{N} - \dot{B})$ (see Fig. 7). Put $P' = (P_1 - (N \cap P_1)) \cup (\dot{N} - \dot{B})$. Then, it is clear that P' is a closed fake surface embedded in the 3-manifold W_1 . Since P_1 expands to $P_1 \cup N$ and $P_1 \cup N$ col-

Polygonal representation of P_1

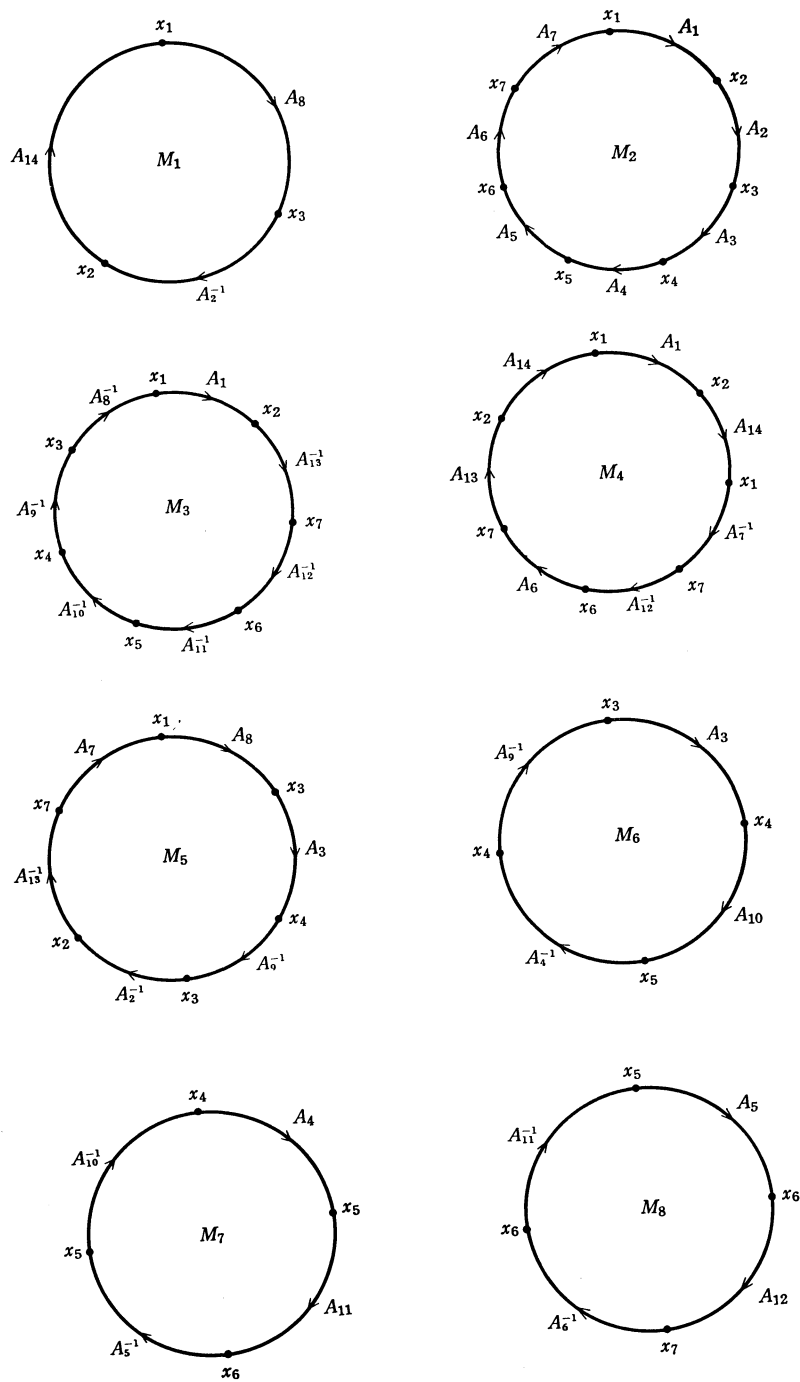


Fig. 5

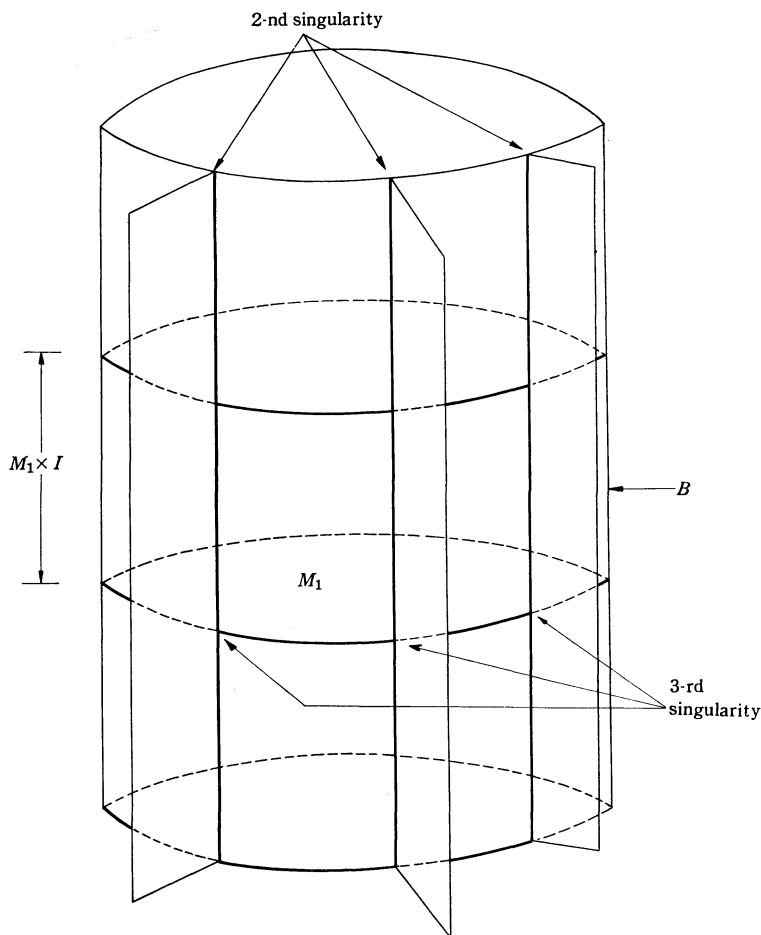


Fig. 6

lapses to P' in W_1 , P_1 and P' belong to the same simple homotopy type in W_1 , that is P' is also a spine of W_1 . By the above construction, the conditions $\#\mathcal{S}_2(P')=1$ and $\#\mathcal{S}_3(P')=6$ are easily seen. Thus, W_1 has a normal spine P' in $\mathcal{D}(1, 6)$.

REMARK. The polygonal representation of P' is shown in Fig. 8. Now, we can prove Theorem 1.

Theorem 1. For the case $1 \leq s \leq 2t-11$ and $t \geq 6$, the set $\mathcal{D}(s, t)$ is non-empty.

Proof. First, it is shown that $\mathcal{D}(1, t)$ is non-empty for $t \geq 6$ by the same argument as that of the proof of Lemma 12 [4], because P' and P_1 belong to

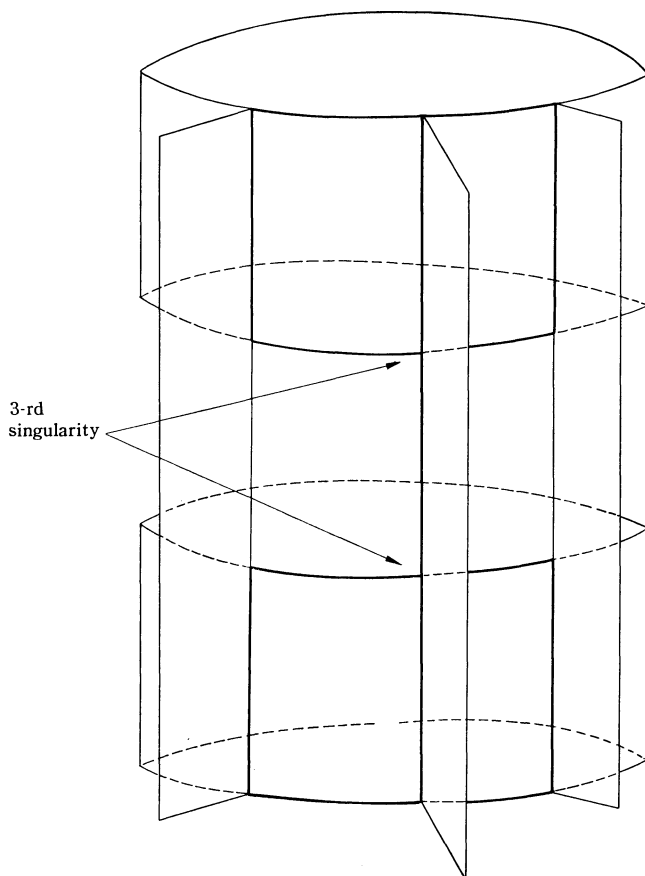


Fig. 7

$\mathcal{D}(1, 6)$ and $\mathcal{D}(1, 7)$, respectively. And, we obtain an element of $\mathcal{D}(s, t)$ with $1 \leq s \leq 2t - 11$ as in the proof of Theorem 6 [4].

4. The Dehn spaces

Let E denote the exterior of a clover knot \mathcal{K} in a 3-sphere Σ , that is, $E = \Sigma - \mathring{N}(\mathcal{K}, \Sigma)$ where $\mathring{N}(\mathcal{K}, \Sigma)$ means the interior of a regular neighborhood $N(\mathcal{K}, \Sigma)$ of \mathcal{K} in Σ . Then, there exists a subpolyhedron F_0 in E which is homeomorphic to F . Of course, F_0 is a spine of E . Regarding the generators S_1 and S_2 of $\pi_1(F)$ as those of $\pi_1(F_0)$, we can write

$$\pi_1(E) = (S_1, S_2: S_1 S_2^{-1} S_1 S_2^2 = 1).$$

Take S_1 and $S_1^{-1} S_2$ as the generators of $\pi_1(E)$, and let i_* denote the homomorphism from $\pi_1(\mathring{E})$ to $\pi_1(E)$ induced by the inclusion map. Since E is an exterior

Polygonal representation of P'

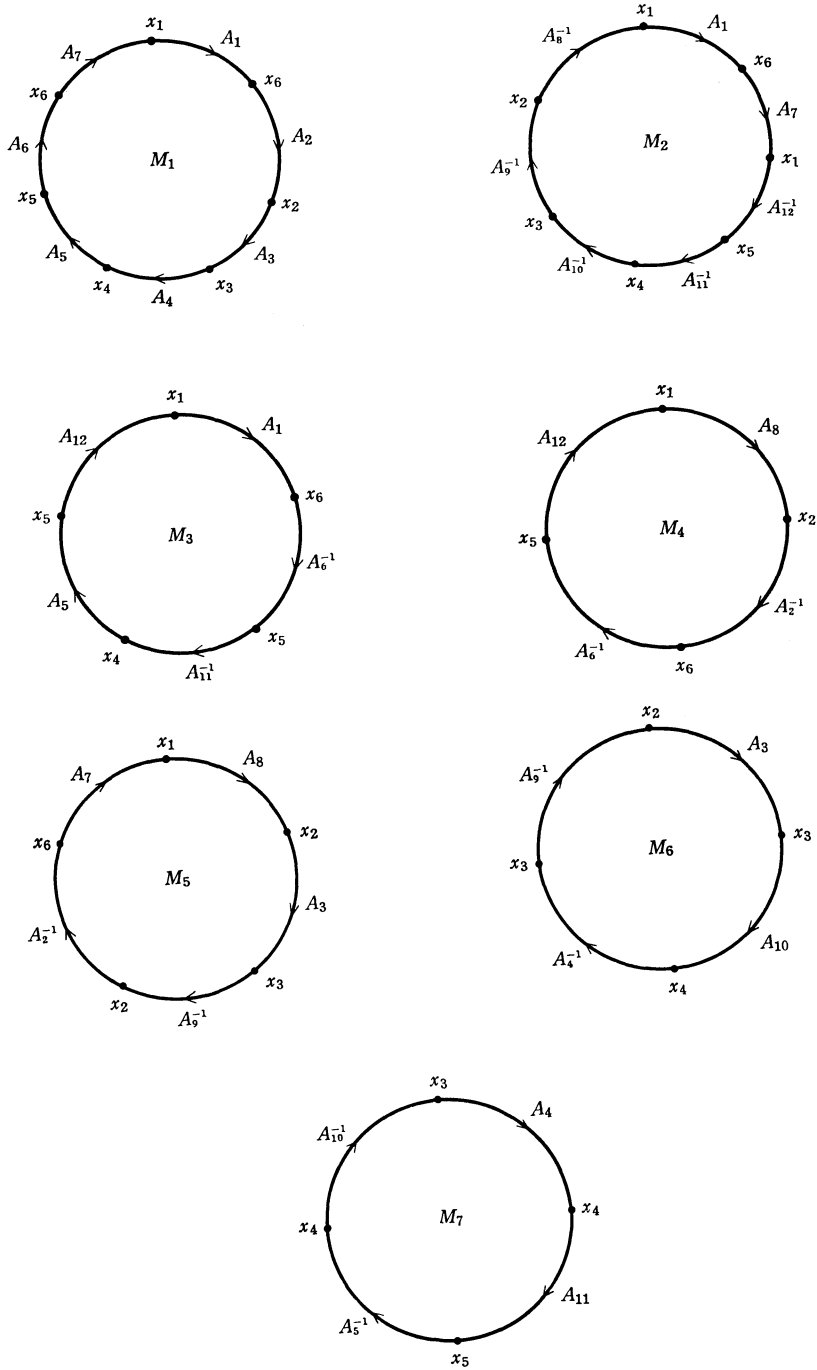


Fig. 8

of a knot, i_* is a monomorphism and we have $i_*^{-1}(S_2^{2k} S_1^{6k-1}) = (S_1^{-1} S_2)^{2k} S_1^{6k-1}$. Let C_k denote the 1-sphere in \dot{E} representing the homotopy class $(S_1^{-1} S_2)^{2k} S_1^{2k-1}$. Note that C_k exists because $2k$ and $6k-1$ are relatively prime.

DEFINITION 2. Define the *Dehn space* V_k of type k to be the 3-manifold obtained from E by attaching a 2-handle along C_k . (Cf. [2])

Theorem 2. Let W_k be the 3-manifold containing P_k as its spine. Then, W_k is the Dehn space of type k .

Proof. By the uniqueness theorem of [1], it is sufficient to prove that the Dehn space V_k contains P_k as its spine, because P_k clearly satisfies the conditions of *standard spine* of [1]. Let N_0 be the 3-rd derived neighborhood of $U(F_0)$ in $E \bmod \dot{U}(F_0)$. We can embed a cylinder $S \times I$ in N_0 in order to satisfy $(S \times I) \cap U(F_0) = S \times 0 = h_k(S)$ and $(S \times I) \cap \dot{N}_0 = S \times 1$ as shown in Fig. 2. Now, let $F_1 = F \cup (S \times I)$ and N_1 the regular neighborhood of F_1 in $E \bmod \dot{F}_1 = S \times 1$. Then, N_1 is homeomorphic to E keeping F_0 fixed, because F_1 collapses to F_0 by collapsing $S \times I$ to $S \times 0$ from $S \times 1$. And hence $S \times 1$ represents the homotopy class $(S_1^{-1} S_2)^{2k} S_2^{6k-1}$ in $\pi_1(N_1)$. Thus, V_k may be regarded as the 3-manifold obtained from N_1 by attaching a 2-handle along $S \times 1$. Then, the 2-handle $B^2 \times I$ collapses to $(\dot{B}^2 \times I) \cup (B^2 \times 1/2)$, where B^2 is a 2-ball and $\dot{B}^2 \times 1/2 = S \times 1$. Thus, V_k collapses to $N_1 \cup (B^2 \times 1/2)$. Since N_1 is a regular neighborhood of F_1 , $N_1 \cup (B^2 \times 1/2)$ collapses to $F_1 \cup (B^2 \times 1/2)$ which is clearly homeomorphic to P_k . Thus, V_k has a spine homeomorphic to P_k . This completes the proof of Theorem 2.

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